AN ANALYTIC SIMILARITY THEORY FOR THE PLANETARY BOUNDARY LAYER STABILIZED BY SURFACE BUOYANCY

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Abstract. An analytic solution for a steady, horizontally homogeneous boundary layer with rotation, f, and surface friction velocity, $\hat{u}_*$, subjected to surface buoyancy characterized by Obukhov length L, is proposed as follows. Nondimensional variables are $\zeta = fz/\eta_* u_*$, $\hat{u} = \eta_* \hat{U}/\hat{u}_*$, $\hat{T} = \hat{\tau}/u_*\hat{u}_*$, where carets denote complex (vector) quantities; $\hat{U}$ is the mean velocity; $\hat{\tau}$ is the kinematic turbulent stress; and $\eta_* = (1 + \xi_N u_*/R_c fL)^{-1/2}$ is a stability parameter. The constant ξ_N is the ratio of the maximum mixing length (l_m) to the PBL depth, u_*/f, for neutrally stable conditions; and R_c (the critical flux Richardson number) is the ratio l_m/L under highly stable conditions. Profiles of stress and velocity in the ocean ($\zeta < 0$) are given by

$$\hat{T} = e^{\hat{\delta}\zeta}$$

$$\hat{u} = \begin{cases} -i\hat{\delta} e^{\hat{\delta}\zeta} & \zeta \leq -\xi_N \\ -i\hat{\delta} e^{-\hat{\delta}\xi_N} - \dfrac{\eta_*}{k}\left[\ln\dfrac{|\zeta|}{\xi_N} + (\hat{\delta} - a)(\zeta + \xi_N) - \dfrac{a}{2}\hat{\delta}(\zeta^2 - \xi_N^2)\right] & \zeta_0 \geq \zeta > -\xi_N \end{cases}$$

where $\hat{\delta} = (i/k\xi_N)^{1/2}$; $a = \eta_*(1/\xi_N + u_*/fLR_c)(1 - \eta_*)$; and ζ_0 is the nondimensional surface roughness. The constants are $R_c = 0.2$ and $\xi_N = 0.052$. The solutions for the atmosphere are similar except $\hat{u}$ is the nondimensional velocity defect and different boundary conditions are reflected by a sign change. The model produces satisfactory predictions of geostrophic drag and near-surface current (wind) profiles under stable stratification.

List of Symbols

A	Rossby similarity parameter
a	$= \beta\mu_*\eta_*$ $= (1-\eta_*)/(\xi_N\eta_*)$
B	Rossby similarity parameter
b	$= (\xi_N/2k)^{1/2}$
f	Coriolis parameter
g	acceleration of gravity
h	$= u_*\eta_*/f$, PBL depth scale
i	$= \sqrt{-1}$
k	von Karman's constant (0.4)
K	eddy viscosity
K_*	$= fK/u_*^2\eta_*^2$, nondimensional eddy viscosity
l	mixing length
l_m	maximum mixing length
L	$= \rho_0 u_*^3(gk\overline{\rho' w'})$, Obukhov length
R_c	critical flux Richardson number (0.2)
Ro_*	$= u_*/f\|z_0\|$, surface friction Rossby number
$\hat{T}$	$= \hat{\tau}/u_*\hat{u}_*$, complex nondimensional stress, magnitude T
t_m	$= \xi_N\eta_*^2/f$, turnover time of largest eddies
$\hat{u}$	$= \eta_*\hat{U}/\hat{u}_*$, complex nondimensional velocity
$\hat{u}_0$	complex nondimensional surface velocity
$\hat{u}_{0N}$	complex nondimensional surface velocity, neutral stability
$\hat{U}$	complex horizontal velocity

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$\hat{U}_0$ complex surface velocity
z vertical co-ordinate
z_0 surface roughness
β $= (1 - \eta_*)(1/R_c + 1/\mu_* \xi_N)$
$\hat{\delta}$ $= (i/k\xi_N)^{1/2}$, complex attenuation coefficient
ζ $= z/h$
ζ_0 $= -|z_0|/h$
η_* $= (1 + \xi_N u_*/R_c f L)^{1/2}$, stability parameter
μ_* $= u_*/fL$
ξ_N dimensionless constant (0.052)
ρ_0 reference fluid density
$\overline{\rho' w'}$ turbulent flux of density variation
$\hat{\tau}$ complex kinematic stress
$\hat{\tau}_0$ $= u_* \hat{u}_*$, complex kinematic surface stress
ϕ $= (kz/u_*)\partial U/\partial z$
ψ $= |z|/L$

1. Background

There is widespread interest in the response of a rotating planetary boundary layer (PBL) to a stabilizing surface buoyancy flux, because such conditions often typify the nocturnal atmospheric boundary layer, especially when there is rapid radiational cooling at ground level. A key element in understanding these stably stratified boundary layers is describing the effect of buoyancy on the exchange of momentum between the free-stream flow (e.g., the geostrophic wind) and the solid surface. Pertinent work on the subject includes that of: Zilitinkevich (1975) and Zilitinkevich and Monin (1974) who matched generalized surface and outer layer regimes using Rossby-similarity theory; Businger and Arya (1974) who developed a steady state, first-order numerical model based on a non-dimensional eddy viscosity derived from the empirical log-linear profile in the stable atmospheric surface layer; and Wyngaard (1975), who solved a time-dependent second-order closure formulation of the developing nocturnal boundary layer.

Data relating surface stress and buoyancy flux to mean wind in the first few tens of meters of the stably stratified atmosphere are fairly plentiful: good examples are from Kansas (Businger *et al.*, 1971) and from Antarctica (Lettau, 1979). By comparison, data relating stable surface conditions to geostrophic wind or wind measured aloft are scarce. Probably the best known are from the Wangara experiment reported by Clarke and Hess (1974); their results have since been corroborated by data from drifting ice stations during the Arctic winter (Leavitt *et al.*, 1978). Arya and Sundararajan (1976) summarize how several similarity theories compare with atmospheric data; and show that the problem is often further complicated by imposition of a length scale associated with the inversion height, which may not depend entirely on local surface conditions.

The diurnally heated oceanic boundary layer, which is also stably stratified, has much in common with the nocturnal atmospheric PBL, but the relationship among surface drift, stress, and buoyancy flux has not drawn much attention, probably because mean surface drift, besides being difficult to measure, is influenced by gravity waves and inertial oscillations. In the ocean, interest has focused instead on the buoyancy flux at the interface between the well mixed layer and the pycnocline. There does exist, however, a fascinating analog between the atmospheric problem and a peculiar regime that may develop under wind-driven pack ice as it drifts across frontal regions marked by abrupt changes in mixed-layer temperature near the ice margin. I have discussed this analog, along with the general similarities between geostrophic drag and sea ice drift, in a recent paper (McPhee, 1981). The purpose of the present note is to present a similarity framework, which I came across during that work, for describing momentum transfer in a steady PBL stabilized by surface buoyancy. I think it demonstrates some perhaps unappreciated connections between Monin–Obukhov surface-layer theory and the structure of turbulence in the outer part of the boundary layer. Using a simple expression for the maximum mixing length attained by turbulent eddies in the PBL, it provides 'universal' analytic solutions for stress and mean current that agree surprisingly well with recent numerical models and with the limited data available.

2. Theory

In a highly idealized version of the sea ice problem described above, we envision a layer of ice with large horizontal extent and uniform surface and salinity characteristics, melting at a constant rate while drifting steadily over a deep, otherwise quiescent ocean. The horizontally homogeneous equation of motion, expressed with complex notation ($i^2 = -1$) is

$$if\hat{U} = \frac{\partial}{\partial z}\hat{\tau} \tag{1}$$

where $\hat{U}$ is the mean horizontal velocity and $\hat{\tau}$ is the horizontal part of the kinematic Reynolds stress tensor. In the atmosphere $\hat{U}$ would be the velocity defect, i.e., the actual wind minus the geostrophic wind. The vertical co-ordinate z is positive upward with the origin at the ice/ocean interface, so that for points in the ocean, z is negative.

The motivation for the similarity theory is to find scales which, when applied to dimensional variables of a whole class of different problems, provide identical profiles of stress and velocity, at least in a region away from the immediate vicinity of the interface. Following arguments set forth by Zilitinkevich (1975), the obvious scale for kinematic stress is the stress acting upon the ocean at its interface with the ice:

$$\hat{\tau}_0 = u_* \hat{u}_*$$

where $\hat{u}_*$ is the (vector) friction velocity in the direction of interfacial stress with magnitude $u_* = (\tau_0)^{1/2}$. (Note that symbols with carets indicate complex [vector] quantities, while those without denote scalar magnitude. A complex quantity divided by $\hat{u}_*$ is oriented such that its real and imaginary components lie respectively parallel and perpendicular to the direction of interfacial stress.) The nondimensional stress is

$$\hat{T} = \hat{\tau}/u_* \hat{u}_*.$$

The nondimensional vertical coordinate is $\zeta = z/h$ where h is the PBL depth scale. From (1) the nondimensional velocity is therefore

$$\hat{u} = \frac{fh\,\hat{U}}{u_* \hat{u}_*}.$$

Differentiation of (1) nondimensionalized yields

$$i\frac{\partial \hat{u}}{\partial \zeta} = \frac{\partial^2 \hat{T}}{\partial \zeta^2}. \tag{2}$$

The stress equation is closed at first order:

$$\hat{\tau} = K\frac{\partial \hat{U}}{\partial z}$$

from which

$$\hat{T} = \frac{K}{fh^2}\frac{\partial \hat{u}}{\partial \zeta} = K_*\frac{\partial \hat{u}}{\partial \zeta}$$

where K_* is the nondimensional eddy viscosity, use of which in (2) yields

$$\frac{\partial^2 \hat{T}}{\partial \zeta^2} = \frac{i}{K_*}\hat{T}. \tag{3}$$

The basic tenet of the present theory is that K_* is determined by the maximum size of turbulent eddies attained within a surface layer which is thin relative to the total PBL depth. Beyond the surface layer, variation in K_* is unimportant. Also, based on numerical studies of the change of stress within the surface layer, it is assumed that the two-layer character of the eddy viscosity is important only with respect to velocity. (In most two-layer formulations, the stress is assumed constant within the surface layer. That is not the case here – for more details see McPhee (1981); but note that while the end results are similar, the approach there is different regarding the surface-layer treatment).

Under these conditions, the stress Equation (3) may be solved treating K_* as constant yielding the solution for stress, given boundary conditions $\hat{T}(0) = 1$, $\hat{T}(-\infty) = 0$,

$$\hat{T} = e^{\hat{\delta}\zeta} \tag{4}$$

where $\hat{\delta} = (i/K_*)^{1/2}$, is a complex coefficient that both attenuates and rotates the stress with increasing depth.

In the outer (Ekman) layer, where K_* is independent of depth for velocity as well as stress, the velocity is given by

$$\hat{u}(\zeta) = -\frac{\partial \hat{T}}{\partial \zeta} = -i\delta\hat{T} \qquad (\zeta < \zeta_m). \tag{5}$$

The maximum eddy viscosity is assumed to be the product of u_*, a maximum characteristic mixing length, l_m, and von Kàrmàn's constant, k,

$$K_* = \frac{k u_* l_m}{f h^2}.$$

The maximum mixing length, l_m, is constrained first by rotation, which imposes a length scale $h = u_*/f$ on the neutrally stable PBL; and second, by gravitational forces associated with surface buoyancy flux, which impose a length scale (Obukhov, 1971).

$$L = \rho_0 u_*^3/(gk\,\overline{\rho' w'})$$

where g is the acceleration of gravity, ρ_0 is the reference density and $\overline{\rho' w'}$ is the turbulent flux of density variation. It is assumed that the interplay of these two scales determines the structure of the outer layer.

Under neutral conditions ($|L| \to \infty$), the expression

$$l_m = \xi_N u_*/f \qquad (L \to \infty) \tag{6}$$

where ξ_N is a dimensionless constant, leads to a Rossby-similarity expression relating stress and surface velocity (McPhee, 1981). There is strong evidence that Rossby-similarity scaling is appropriate for the neutral ice-ocean PBL, both from measurements of surface drift and stress during the AIDJEX main experiment (*ibid*) and from measurements of mean and turbulent velocity throughout the under-ice PBL (McPhee and Smith, 1976).

When surface buoyancy is strongly stabilizing (L small and positive), the scale of turbulence is limited by the work required to displace fluid vertically. Following, e.g., the reasoning of Businger and Arya (1974), the mixing length must then scale with L, i.e., $l = c_1 L$. Zilitinkevich (1975) has shown that the proportionality constant may be evaluated as follows: We have

$$c_1 = \frac{l}{L} = \frac{K}{u_* k L}.$$

If l is small compared to the PBL scale depth, which it certainly is under very stable conditions, then stress is nearly constant in a layer of thickness of order l, so

$$c_1 = \frac{u_*}{k\dfrac{\partial U}{\partial z}L} = \left(g\frac{\overline{\rho' w'}}{\rho_0}\right)\Bigg/\left(u_*^2\frac{\partial U}{\partial z}\right) = R_f$$

where R_f is the flux Richardson number. It follows that the maximum mixing length under highly stable conditions is given by

$$l_m = R_c L \qquad (L \to 0^+) \tag{7}$$

where R_c is the maximum (critical) flux Richardson number. While values reported for R_c vary, estimates tend to be near $R_c = 0.2$ (Tennekes and Lumley, 1972, p. 98).

The simplest expression for l_m including both limits expressed by (6) and (7) is

$$l_m = \frac{u_* \xi_N}{f}\left(1 + \frac{\xi_N}{R_c}\frac{u_*}{fL}\right)^{-1}. \tag{8}$$

The similarity hypothesis is that universal scales exist for the free turbulence of the outer layer (i.e., away from the immediate effects of the solid surface), which requires the maximum eddy viscosity to be independent of other variables. For the neutral case, $h = u_*/f$ and $K_* = k\,\xi_N$; therefore

$$K_* = \frac{ku_*^2}{f^2h^2}\xi_N\left(1 + \frac{\xi_N \mu_*}{R_c}\right)^{-1} = k\xi_N \tag{9}$$

where $\mu_* = u_*/fL$. Solving for h, and defining a stability parameter, $\eta_* = (1 + \xi_N\mu_*/R_c)^{-1/2}$, yields the following scales.

Depth:	$u_*\eta_*/f$
Velocity:	$\hat{u}_*/\eta_*$
Eddy Viscosity:	$u_*^2\eta_*^2/f$.

Equations (4) and (5) specify the PBL stress and velocity for $\zeta < \zeta_m$, where $\zeta = fz/u_*\eta_*$, $\delta = (i/k\xi_N)^{1/2}$, and ζ_m is the nondimensional level marking the extent of the surface layer. Given R_c, ξ_N is the only 'free' constant in the theory and remains to be evaluated. R

Under neutral conditions ($\eta_* = 1$), $\zeta_m = -\xi_N$ is the surface-layer depth and we assume that within the surface layer, the mixing length is proportional to the distance from the surface. The nondimensional velocity gradient in the surface layer is thus

$$\frac{\partial \hat{u}}{\partial \zeta} = \frac{\hat{T}}{K_*} = \frac{\hat{T}}{-k\zeta} = \left(\frac{1 + \delta\zeta}{-k\zeta}\right)(\zeta > \zeta_m, \mu_* = 0) \tag{10}$$

where a Taylor-series expansion replaces $\hat{T}$ for $|\zeta|$ small. Integration of (10) from the level ζ_m yields

$$\hat{u}(\zeta) - \hat{u}(\zeta_m) = \frac{-1}{k}\left[\ln\frac{\zeta}{\zeta_m} + \delta(\zeta - \zeta_m)\right]. \tag{11}$$

Because of the logarithmic nature of (11), the surface velocity is evaluated at a nondimensional level, $\zeta_0 = -f|z_0|/u_*\eta_*$, where z_0 is a surface roughness scale. (Note that for neutral conditions $|\zeta_0|$ is the inverse of the surface-friction Rossby number, $\mathrm{Ro}_* = u_*/fz_0$). The surface velocity is

$$\hat{u}_{0N} = -i\delta e^{-\delta\xi_N} - \frac{1}{k}\left[\ln\frac{\zeta_0}{\zeta_m} + \delta\xi_N\right] \tag{12}$$

using the approximation $\delta(\zeta_0 - \zeta_m) \simeq -\delta\zeta_m = \delta\xi_N$, since $|\zeta_0|$ is much smaller than ξ_N. It is clear that the imaginary part of (12) is independent of the surface roughness, and is given by

$$\mathrm{Im}\,(\hat{u}_{0N}) = -b\left[\frac{e^{-b}}{\xi_N}(\cos b + \sin b) + \frac{1}{k}\right] \tag{13}$$

where $b = (\xi_N/2k)^{1/2}$. During the summer drift of the AIDJEX ice stations, surface velocity and interfacial stress were monitored independently (McPhee, 1981). Average conditions were found to be approximately: $u_* = 1$ cm s^{-1}, $U_0 = 13$ cm s^{-1}, with an angle of about 24° between stress and surface velocity. Melting was slow enough to assume neutral conditions, so $\mathrm{Im}\,(\hat{u}_{0N}) \simeq -13 \sin 24°$, which provides the value, $\xi_N = 0.052$.

Equations (4) and (5) with $\xi_N = 0.052$ are plotted in Figure 1 along with solutions for neutral conditions from the numerical model of Businger and Arya (1974). The analytic solution depends only on the parameter ξ_N, since velocity in the surface layer is not considered.

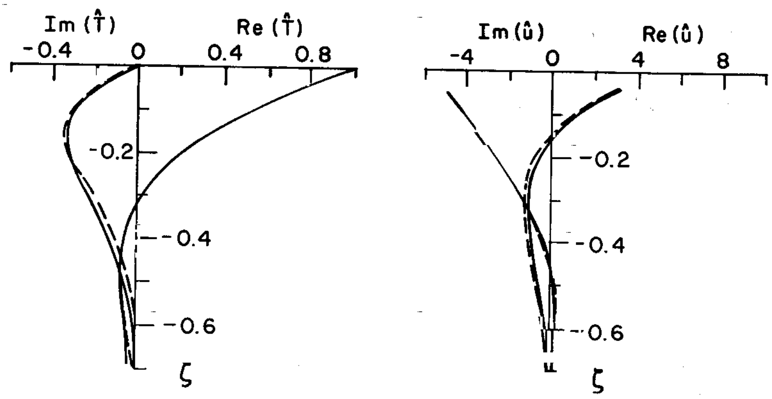


Fig. 1. Nondimensional stress and velocity profiles for $\xi_N = 0.052$ (solid). Dashed curves are from Businger and Arya (1974) for neutral stability in which case vertical co-ordinates for each model are identical (fz/u_*). Velocity profiles in surface layer $(|\zeta| < \xi_N)$ depend on ζ_0, and are not shown.

The complete theory requires a formulation for velocity in the surface layer. In the earlier work on drift at the ice margin (McPhee, 1981) under conditions of varying stability, I used an approach in which the surface-layer mixing length increased linearly with depth until it reached the maximum value given by (8). This required the nondimensional surface-layer thickness (ζ_m) to vary as the stability factor, η_*. At high stabilities, $|\zeta_m|$ becomes quite small and one can show, for example, that standard instrument heights in the atmosphere are beyond the logarithmic surface layer. From this point of view, measurements made under stable conditions are often outside the surface layer, despite the fact that stress attenuation may be small and there is relatively little deflection of the wind profile. That approach led to plausible predictions of surface velocity values (or velocity defect), as deduced from published measurements of geostrophic drag (see Figure 10 of McPhee, 1981).

In the present work, a slightly different conceptual approach is introduced. Its impact on the end results for geostrophic drag and even near-surface currents is slight, but it demonstrates, I think, some interesting connections between so-called surface-layer theory and turbulence in the outer layer. The idea is that the nondimensional surface-layer depth is fixed at its neutral value ($\zeta_m = -\xi_N$), and the surface-layer eddy viscosity takes a form associated with the log-linear profile; i.e.,

$$K = \frac{-kzu_*}{1 + \beta(|z|/L)}, \qquad -z_0 > z > \frac{-u_*\eta_*\xi_N}{f}. \tag{14}$$

The nondimensional form of (14) is

$$K_* = \frac{-k\zeta}{\eta_*}(1 - \beta\,\mu_*\eta_*\zeta)^{-1}$$

and must match the outer-layer eddy viscosity at $\zeta_m = -\xi_N$. This provides an expression for β, given after some manipulation, by

$$\beta = \left(\frac{1}{R_c} + \frac{1}{\mu_*\xi_N}\right)(1 - \eta_*). \tag{15}$$

Note that β is not prescribed as an external factor, but instead is calculated as a weak function of μ_*.

Velocity in the surface layer is found in the same way as for the neutral case, by integrating the equation

$$\frac{\partial \hat{u}}{\partial \zeta} = \frac{\hat{T}}{K_*} \simeq \frac{-\eta_*}{k\zeta}(1 - \beta\mu_*\eta_*\zeta)(1 + \delta\zeta).$$

This leads to the following description for the entire boundary layer:

$$\hat{T} = e^{\delta\zeta} \tag{16}$$

$$\hat{u}(\zeta) = \begin{cases} -i\delta e^{\delta\zeta} & \zeta \le \zeta_m \\ \hat{u}(\zeta_m) - \dfrac{\eta_*}{k}\left[\ln\dfrac{|\zeta|}{\xi_N} + (\delta - a)(\zeta + \xi_N) - \dfrac{a}{2}\delta(\zeta^2 - \xi_N^2)\right], & \zeta > \zeta_m \end{cases} \tag{17}$$

with $a = \beta\mu_*\eta_*$ and $\zeta_m = -\xi_N$. The nondimensional surface velocity is

$$\hat{u}_0 = \hat{u}(\zeta_m) - \frac{\eta_*}{k}\left[\ln\frac{|\zeta_0|}{\xi_N} + (\delta - a)\,\xi_N + \frac{a}{2}\delta\xi_N^2\right]. \tag{18}$$

Equations (16) and (17), with the parameters $R_c = 0.2$ and $\xi_N = 0.052$, constitute the similarity theory. The 'geostrophic' drag law, i.e.,

$$\hat{u}_0 = \eta_*\frac{\hat{U}_0}{\hat{u}_*}$$

is given by (18).

3. Discussion

Similarity profiles and hodographs for three values of μ_* are shown in Figure 2, where surface characteristics were determined according to $u_* = 1$ cm s^{-1}, $z_0 = 5$ cm, $f = 1.4 \times 10^{-4}$ s^{-1}, which are representative for sea ice. In order to maintain $\mu_* = 50$ under such conditions requires a melting rate of order 1 m of ice per day. Profiles in dimensional coordinates are shown in Figure 3; dimensional kinematic stress is shown in Figure 4. The depth of the boundary layer is seen to decrease substantially according to the scaling

$$h = \frac{u_* \eta_*}{f}.$$

Note that the actual PBL depth is about half of h. For $\mu_* = 50$, η_* is 0.27, so that the PBL is only about a quarter as thick as it would be with no surface buoyancy. For μ_* large, the PBL depth varies approximately as $\mu_*^{-1/2}$, in agreement with Wyngaard (1975), Businger and Arya (1974), and Zilitinkevich (1975). However, the magnitude of Wyngaard's (1975) PBL depth is considerably smaller than 0.5 h,

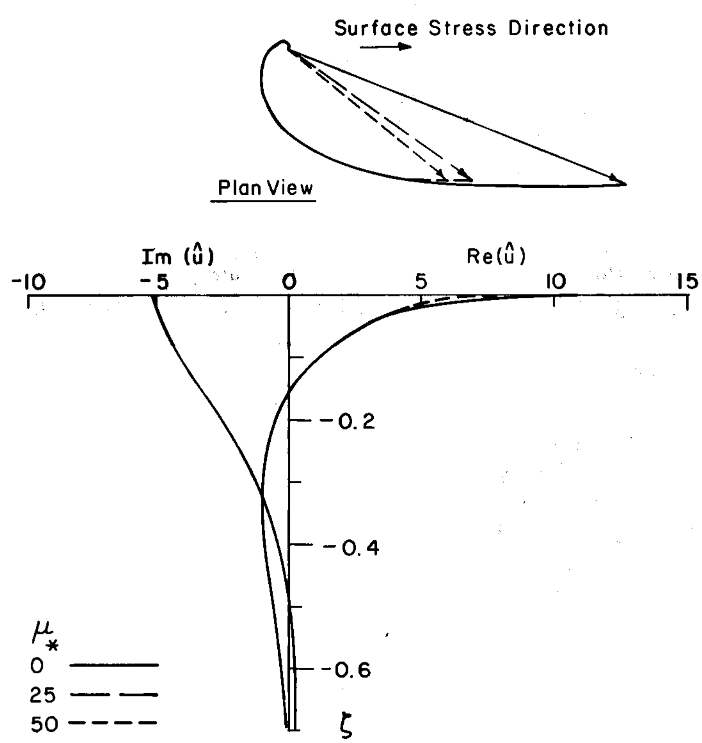
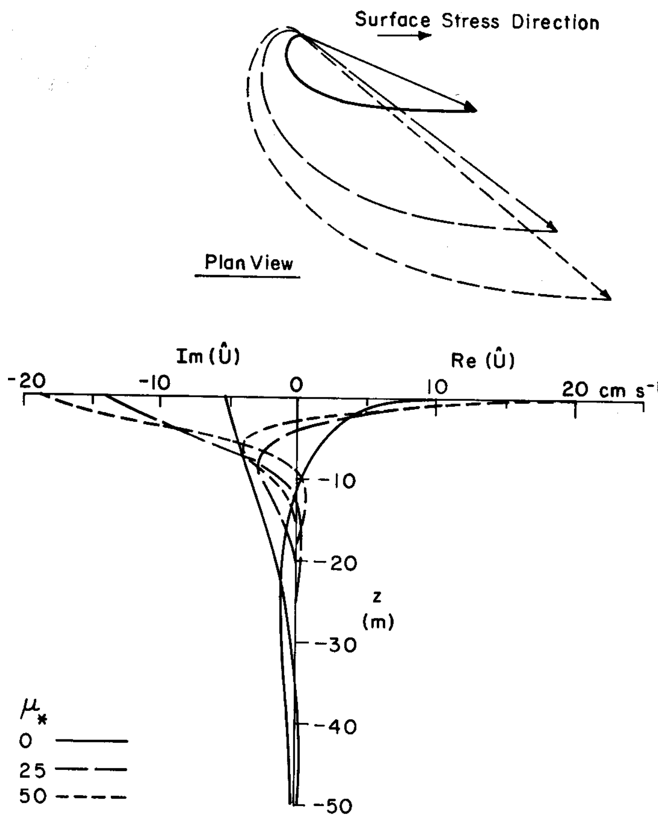


Fig. 2. Nondimensional profiles and hodographs for three values of μ_*, showing impact of stability on surface characteristics when other external parameters are $u_* = 1$ cm s^{-1}, $z_0 = 5$ cm, $f = 1.4 \times 10^{-4}$ s^{-1}. Note that depth of point on hodograph may be determined by dropping vertical to corresponding point on 'real' velocity profile.



nsional velocity components corresponding to Figure 2 where real axis is aligned with $\hat{u}_*$.

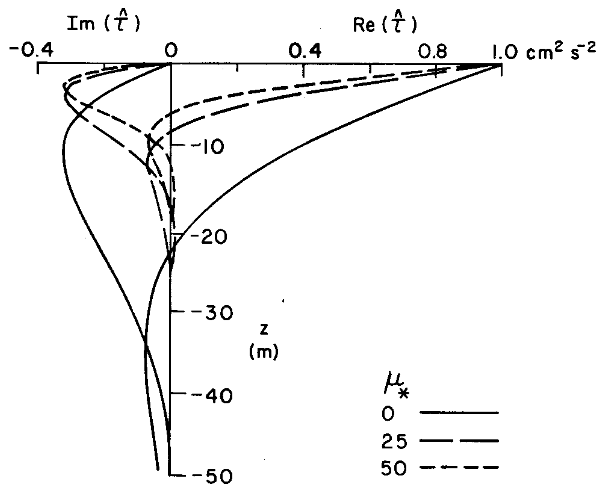


Fig. 4. As in Figure 3, except profiles for kinematic stress.

which compresses, e.g., his equilibrium stress profiles relative to the present work (although the shapes are roughly similar). As Figure 1 shows, the agreement with Businger and Arya's (1974) profiles is excellent, given that their PBL depth also falls off approximately as $\mu_*^{-1/2}$.

The usual way of expressing geostrophic drag is via the Rossby-similarity functions A and B, given by

$$B = k\,\mathrm{Im}\left(\frac{\hat{U}_0}{\hat{u}_*}\right) = \frac{k}{\eta_*}\,\mathrm{Im}\,(\hat{u}_0)$$

$$A = \ln(\mathrm{Ro}_*) - k\mathrm{Re}\left(\frac{\hat{U}_0}{\hat{u}_*}\right) = \ln(\mathrm{Ro}_*) - \frac{k}{\eta_*}\,\mathrm{Re}(\hat{u}_0)$$

where $\hat{u}_0$ is given by (18).

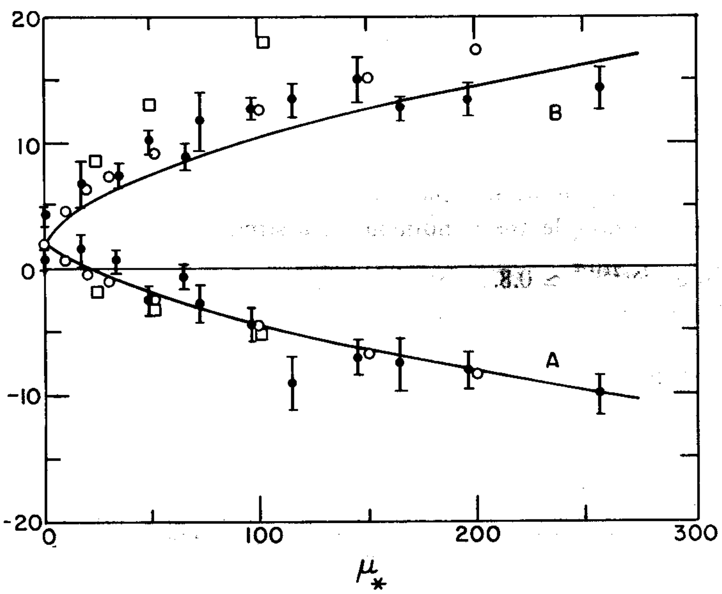


Fig. 5. The Rossby-similarity parameters A and B as functions of μ_*. Solid curves are from present theory; data points with error bars are from Clarke and Hess (1974); open circles and squares are, respectively, from numerical models of Businger and Arya (1974) and Wyngaard (1975).

Figure 5 shows the functions A and B compared with the data of Clarke and Hess (1974) and some values from the models of Businger and Arya (1974) and Wyngaard (1975). Considering the scatter in the data (see McPhee [1981], for a discussion of the effect of uncertainty in u_* on estimates of μ_* and B), this result is encouraging confirmation that plausible similarity scales have been chosen.

In the near-surface layer of the atmosphere, Monin–Obukhov similarity theory holds that the nondimensional wind shear (and mean gradients of other pertinent properties) may be expressed as universal functions of height divided by the Obukhov length, i.e.,

$$\phi = \frac{kz}{u_*}\frac{\partial U}{\partial z} = \phi(\psi)$$

where $\psi = |z|/L$. Businger *et al.* (1971) found their data (for ψ positive) to be well fitted by the straight line

$$\phi = 1 + 4.7\psi. \tag{19}$$

The first-order numerical model of Businger and Arya (1974), to which the present work is closely related, utilizes a K-distribution which provides (19) in the near-surface limit. A careful study by Lettau (1979) of near-surface profiles over the Antarctic ice cap led him to a slightly different expression:

$$\phi = (1 + 5\psi)^{3/4}. \tag{20}$$

Note that in (20) the slope of ϕ (i.e., $\partial\phi/\partial\psi$) decreases with increasing ψ.

In the present work, the point of view differs somewhat from Monin–Obukhov theory in that the rotational scale, u_*/f, is allowed to play a role throughout the entire PBL. In the expression for eddy viscosity in the region affected by the interface, β is calculated from other prescribed parameters, namely R_c, ξ_N, and μ_*. From (15), it is clear that β reaches a constant value $1/R_c$ only in the limit $\mu_* \to \infty$. In addition, ϕ also depends on the attenuation of stress in the surface layer; at the level $z_m = -\eta_* u_* \xi_N/f$, for example, the nondimensional stress magnitude is

$$T = e^{-(\xi_N/2k)^{1/2}} \simeq 0.8.$$

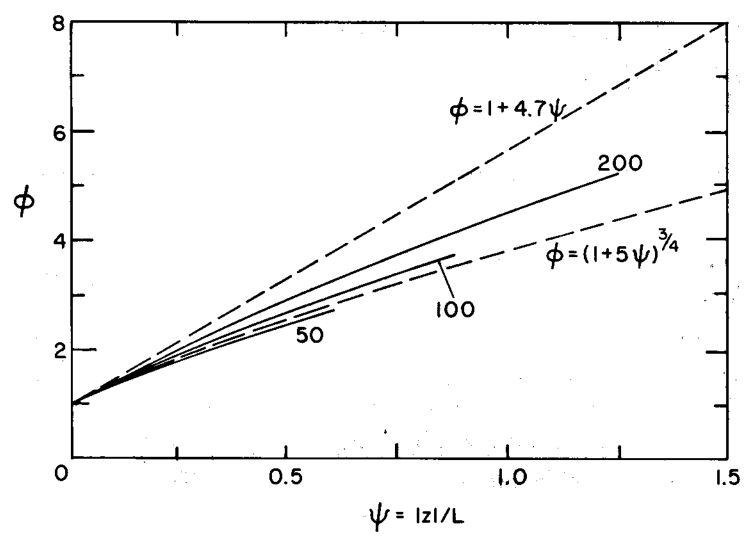


Fig. 6. Magnitude of dimensionless, surface-layer wind shear, $\phi = (kz/u_*)\partial U/\partial z$, for three values of $\mu_* = u_*/fL$ plotted as function of $\psi = |z|/L$. Theoretical curves are limited to values of $\psi = -\eta_* u_* \zeta$, such that the nondimensional stress magnitude is greater than 0.8. Dashed curves are empirical data fits: $\phi = 1 + 4.7\psi$ (Businger *et al.*, 1971); and $\phi = (1 + 5\psi)^{3/4}$ (Lettau, 1979).

For concreteness, in a midlatitude, atmospheric boundary layer with $u_* = 20\ \mathrm{cm\,s^{-1}}$ and $\mu_* = 100$, the stress would be about 20% less than its surface value at a height of 20 m.

Figure 6 shows the magnitude of ϕ calculated for three values of μ_* from the expression

$$\hat{\phi} = \frac{-k\zeta}{\eta_*}\frac{\partial \hat{u}}{\partial \zeta} = \hat{T}(1 - \beta\eta_*\mu_*\zeta)$$

where the Taylor expansion for $\hat{T}$ and the identity $\psi = -\eta_*\mu_*\zeta$ have been used to provide

$$\hat{\phi} = (1 + \beta\psi)\left(1 - \frac{\delta}{\eta_*\mu_*}\psi\right).$$

The curves are limited to values of ψ such that $T \geq 0.8$. Also shown are the empirical formulas, (19) and (20). Note that the only external parameters for the theoretical curves are R_c ($= 0.2$) and ξ_N ($= 0.052$ as determined from the drift of sea ice).

4. Summary

The ideas outlined above are an attempt to combine a simplied K-theory similar in some respects to that of Businger and Arya (1974), with scaling arguments set forth by Zilitinkevich (1975). The main assumptions are:

(1) There is one dominant mixing length, determined by f, u_*, and L, that characterizes turbulence throughout most of the PBL. Heuristic arguments suggest that its magnitude ranges from $\xi_N u_*/f$ under neutrally stable conditions to $R_c L$ under conditions of high stability.

(2) In a thin surface layer where $|z| < \xi_N h$ (about 10% of the total PBL depth), the mixing length is limited by the distance from the surface. This is important only insofar as it affects fluid velocity within the surface layer; turbulent stress, and mean velocity beyond the surface layer, are specified as simple analytic functions of u_*, f, and L.

(3) In the surface layer itself, the velocity profile is assumed to be nearly log-linear (it would be exactly if stress were constant). The link to Monin–Obukhov surface-layer theory comes through the parameter β given by (15), which shows its dependence on μ_*, the ratio of the neutral PBL depth to the Obukhov length. The dependence implies that the dimensionless shear, ϕ, is not a unique function of ψ in contradiction to Monin-Obukhov theory; however, the theoretical variation is not large (see Figure 6), certainly no larger than experimental scatter.

Two criticisms of the model developed here are apparent. First, the PBL is rarely steady. The very presence of surface buoyancy means time changes in density structure, and constant surface stress is not often maintained for extended periods. But in this regard it is useful to consider the turnover time of the largest eddies, which is a measure of how rapidly kinetic energy of the mean flow is lost to turbulence, and is estimated (see Tennekes and Lumley, 1972) by

$$t_m = l_m/u_* = \xi_N \eta_*^2/f.$$

For neutral conditions, t_m is of order 5–6 min in agreement with under-ice observations (McPhee and Smith, 1976), but for $\mu_* = 50$, t_m is less than 30 s; thus one may infer that time scales for turbulence are often considerably shorter than those for the driving forces. However, it should also be noted that rapid temporal changes usually imply sharp horizontal gradients, which may also affect the local momentum balance. Inertial oscillations are expected in the ocean, but observations and numerical work (McPhee, 1980) indicate that their impact on turbulent stress levels and near-surface shear is not large.

The second obvious drawback is that no account is taken of density changes deeper in the layer. If the pycnocline depth (inversion height) is much less than, say 0.5 h, then turbulence, especially in the outer parts of the PBL, will be affected in a way that does not fit into the simple similarity framework. Nevertheless, to the extent that the concepts outlined here successfully describe the effect of surface buoyancy, they point toward a quite simple approach to treating turbulence near a pycnocline interface, which is a subject for further work.

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